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SPLIT GRAPHS OF DILWORTH NUMBER 2

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1. Introduction

The purpose of this article is to give a simple proof that the class of split graphs with Dilworth number at most 2 coincides with the class of graphs G such that G and its complement $\bar{G}$ are interval graphs. A forbidden subgraph characterization of this class is also presented. This result is also given in [1] with a much longer and more involved proof. Graphs of Dilworth number 2 have been studied in [2] where they are called threshold signed graphs. A linear-time algorithm is presented to recognize threshold signed graphs.

2. Split graphs and Dilworth number

We first recall some definitions and results which are needed to characterize split graphs of Dilworth number at most 2. All graph theoretical terms not defined here can be found in [7].

G is a *split graph* if its node set may be partitioned into a clique and a stable set. A graph is *triangulated* (or *chordal*) if every cycle of length strictly greater than 3 possesses a chord.

Theorem 1 (Foldes and Hammer [4]). *The following statements are equivalent for a graph G:*

(1) *G is a split graph:*

(2) *G and $\bar{G}$ are triangulated:*

(3) *G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

Theorem 2 (Gilmore and Hoffman [6]). *The following statements are equivalent for* *a graph G:*

(1) *G is an interval graph:*

(2) *G is triangulated and $\bar{G}$ is a comparability graph;*

(3) *The maximal cliques of G can be numbered $K_1, \ldots, K_m$ in such a way that for each node x, $x \in K_i \cap K_j$ $(i<j)$ implies $x \in K_k$ for all k, $i<k<j$.*

G is a *threshold graph* if one can associate weights a_i with the nodes and a threshold value S such that a set of nodes in G is stable iff the sum of its weights is at most S.

A *preorder* is a transitive and reflexive relation. Here we define the *vicinal preorder* on the node set of a graph as follows:

$$x \leqslant y \quad \text{if } N(x) \subseteq N(y) \cup \{y\}$$

where $N(x)$ is the set of neighbours of x, i.e. the set of nodes y linked to x.

The *Dilworth number* of G is the maximum integer k such that there exist in G k mutually incomparable nodes with respect to $\leqslant$. From the theorem of Dilworth this is also the smallest k for which there exists a partition of the node set into k chains (sets of mutually comparable nodes).

Theorem 3 (Chvátal and Hammer [3]). *The following statements are equivalent for a graph G:*

(1) *G is a threshold graph;*

(2) *The Dilworth number of G is* 1;

(3) *G does not contain any induced subgraph isomorphic to $2K_2$, C_4 or P_4.*

G is a *threshold signed graph* (TS-graph) if one can associate weights a_i with the nodes and threshold values S, T such that $[i, j]$ is an edge iff $|a_i + a_j| \geqslant S$ or $|a_i - a_j| \geqslant T$.

Theorem 4 (Benzaken, Hammer and de Werra [2]). *The following statements are equivalent for a graph G:*

(1) *G is a TS-graph;*

(2) *The Dilworth number of G is at most* 2;

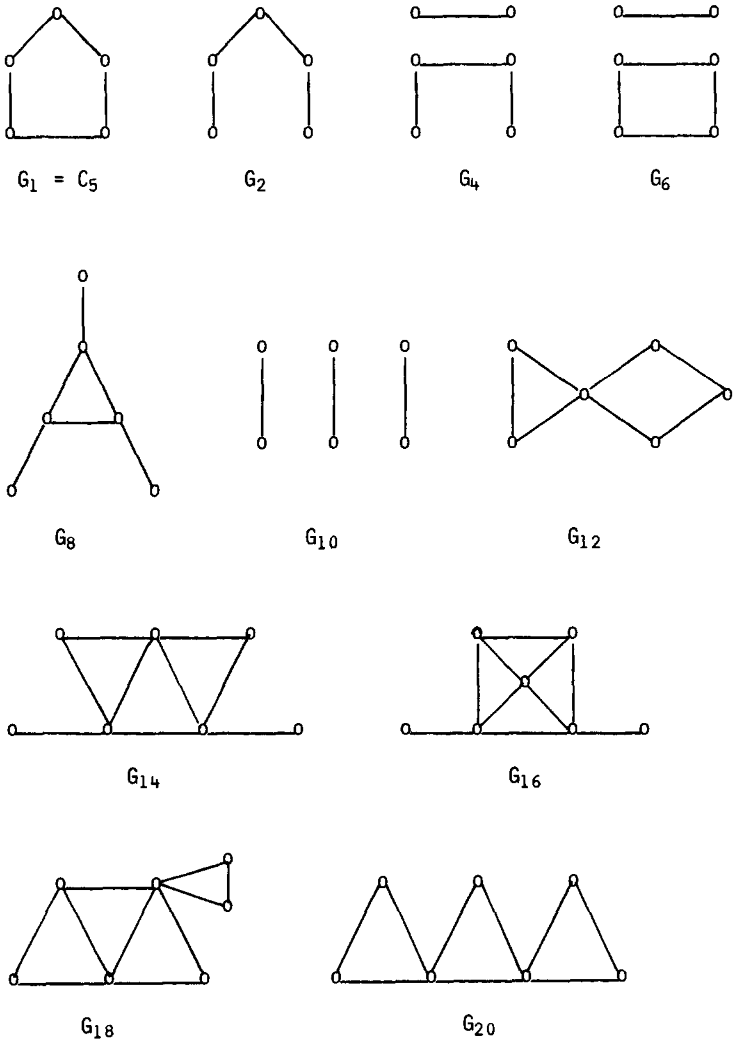
(3) *G does not contain any of the graphs $G_1, \ldots, G_{21}$ in Fig.* 1 *as an induced subgraph.*

Remark. It is also known that a TS-graph is a comparability graph; furthermore the complement of a TS-graph is also a TS-graph.

Theorem 5 (Foldes and Hammer [5]). *The following statements are equivalent for a split graph:*

(1) *G is an interval graph;*

(2) *The Dilworth number of G is at most* 2.



$G_{2i+1} = \bar{G}_{2i}\,,\quad i = 1,2,\ldots,10$

Fig. 1.

We can now state the main result. The equivalence of (1) and (3) is also given in [1] with a much longer and involved proof.

Proposition 1. *Let G be a graph; the following statements are equivalent:*

(1) *G and $\bar{G}$ are interval graphs;*
(2) *G is a split TS-graph;*
(3) *G contains none of the induced subgraphs of Fig. 2.*

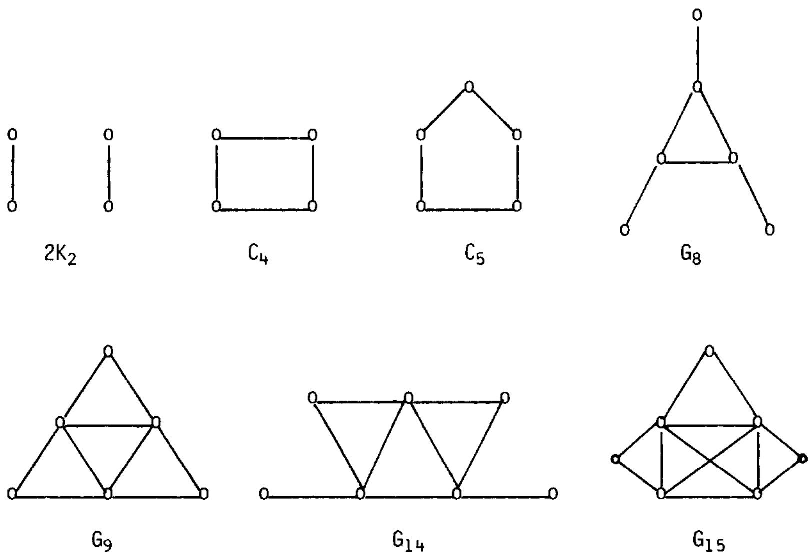


Fig. 2.

Proof. (1) ⇒ (2): From Theorem 2, G and $\bar{G}$ are triangulated graphs. Hence from Theorem 1, G is a split graph. From Theorem 5 and Theorem 4, it follows that T is a TS-graph.

(2) ⇒ (1): This follows immediately from Theorems 4 and 5 and from the fact that $\bar{G}$ is also a split TS-graph.

(2) ⇒ (3): Since G is a split graph, it does not contain $2K_2$, C_4 or C_5 according to Theorem 1. Since G is a TS-graph it does not contain G_8, G_9, G_{14} or G_{15} from Theorem 4.

(3) ⇒ (2): Since G does not contain $2K_2$, C_4 or C_5, it is a split graph; furthermore since G contains neither $2K_2$ nor C_4, the only induced subgraphs of Fig. 1 which it may contain are G_8, G_9, G_{14} and G_{15}. Since by assumption G does not contain them, G is a TS-graph.

Remark. The class described in Proposition 1 includes properly the class of threshold graphs.

3. Recognizing threshold signed graphs

TS-graphs can be recognized in time $O(n+m)$ where n is the number of nodes and m is the number of edges with the following straightforward algorithm. We assume that the graph is given by the list of sets $N(v)$ of neighbours of each node v. Furthermore the nodes are ordered according to nonincreasing degrees. Each node i will receive a label $l(i)=L$ (left set) or R (right set).

Algorithm.

Step 1. Scan the list of nodes until one gets a node k such that $N(k) \not\subseteq N(k-1) \cup \{k-1\}$; if such a node has been found, set $l(i) = L$ for $i = 1, \ldots, k-1$; $l(k) = R$ and go to Step 2. If no k is found, (*G is a threshold graph *). Stop.

Step 2. Let last $= k-1$ (* this is the last node with label opposite to the label of the current node k*). For $i = k+1, \ldots, n$ repeat the following sequence: if $N(i) \subseteq N(i-1) \cup \{i-1\}$, then $l(i) = l(i-1)$; otherwise if $N(i) \subseteq N(\text{last}) \cup \{\text{last}\}$, then $l(i) = l(\text{last})$, last $= i-1$, else (* i.e. $N(i) \not\subseteq N(\text{last}) \cup \{\text{last}\}$, *) G is not a TS-graph. Stop.

Complexity. In Steps 1 and 2 we have to check whether for a given pair j, i of nodes we have $N(i) \subseteq N(j) \cup \{j\}$. For this purpose we may use a vector MV of length n which is initialized to 0. When a node j has received a label, then for all nodes p in $N(j) \cup \{j\}$, we set $\mathrm{MV}(p) = j$; the other entries need not be defined. The time to do this is $O(d(j))$. For examining whether $N(i) \subseteq N(j) \cup \{j\}$, we have to check the entries $\mathrm{MV}(p)$ for all p in $N(i)$; this can be done in $O(d(i))$.

In Steps 1 and 2 of the algorithm one sees that for any node k the vector MV has to be reset for k at most 2 times. Thus the total time for the algorithm is $O(n + \text{sum of degrees } d(i))$, i.e. $O(n+m)$.

Acknowledgment

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